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# ON THE THEORY OF STABILITY OF CYLINDRICAL THIN-WALLED SHELLS UNDER THE ACTION OF EXTERNAL PRESSURE*

A.N. GUZ'

A method is described of constructing a linearized, two dimensional theory of stability of elastic, thin-walled circular cylindrical shells acted upon by a uniform external pressure in the form of a "follower" load. The relations of the theory of shells constructed with the help of the Kirchhoff—Love hypothesis for the case when the subcritical deformations are small and the subcritical state is determined according to the geometrically linear theory, are used. The "follower" load is determined in the above formulation using the refined expressions given in /1,2/. As a result, a fundamental system of differential equations with a symmetric operator matrix is obtained for the problem in question. A characteristic equation is obtained for a hinged shell in a subcritical membrane state. Asymptotic analysis of the roots of this equation yields the conditions under which the solutions for the case of external pressure in the form of a follower and a dead load coincide.

We note that a solution of the linearized problem of stability of a hinged cylindrical shell acted upon by a uniform external pressure in the form of a dead load was already obtained by Mises in /3/. The solution given in /3/ is quoted in a number of monographs, in particular in /4/, and is widely used together with its various generalizations, in the analysis of the theoretical and experimental results. A majority of the experimental results however is obtained under the conditions when the uniform external pressure is realized by means of hydrostatic loading, i.e. in the form of a follower load.

**1. Formulation of the problem.** We consider a circular cylindrical shell of constant thickness $h$, radius $R$ and length $L$, under a uniform external load of intensity $q$. The shell material is assumed elastic and isotropic. We construct a fundamental system of differential equations for the case when the external pressure has the form of a follower load. We adopt the $(x, y, z)$-coordinate system defined in ch.2 of /4/ and choose the corresponding positive directions of some of the magnitudes. We write the linearized equations of the theory of shells in the form

$$\mathbf{LU} + \mathbf{BU} + \mathbf{Q}(\mathbf{q} - \rho h \mathbf{U}^{\cdot\cdot}) = 0 \tag{1.1}$$

$$Q_{ij} = \frac{1-\nu^2}{Eh}(\delta_i^1\delta_j^1 + \delta_i^2\delta_j^2 - \delta_i^3\delta_j^3)$$

$$L_{11} = \frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial y^2}; \quad L_{12} \equiv L_{21} = \frac{1+\nu}{2}\frac{\partial^2}{\partial x\,\partial y}$$

$$L_{13} \equiv L_{31} = -\frac{\nu}{R}\frac{\partial}{\partial x}; \quad L_{22} = \left(1 + \frac{h^2}{12R^2}\right)\left(\frac{\partial^2}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x^2}\right)$$

$$L_{23} \equiv L_{32} = \left[-\frac{1}{R} + \frac{h^2}{12R}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\frac{\partial}{\partial y}$$

$$L_{33} = \frac{h^2}{12}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 + \frac{1}{R^2}$$

Here $\mathbf{U}$ is the middle surface displacement vector with components $u$, $v$ and $w$, $\mathbf{q}$ is the surface load vector with components $q_x, q_y$ and $q_z$, $\mathbf{L}$ is the symmetric matrix of differential operators of the linear theory of shells, $\mathbf{B}$ is the matrix of differential operators with parametric terms, obtained as a result of linearizing the nonlinear equations, $\nu$ is the Poisson's ratio, $E$ is the Young's modulus, $\delta_i$ is the Kronecker delta and $\rho$ is the shell material density. In the simplest case of the subcritical membrane state, the elements of the matrix $\mathbf{B}$ have the form /4/

$$B_{ij} = \delta_i^3\delta_j^3 qR\frac{1-\nu^2}{Eh}\frac{\partial^2}{\partial y^2} \tag{1.2}$$

In order to construct the fundamental system of equations in the above formulation we must obtain, in terms of the displacements, the components of the surface load corresponding

*Prikl.Matem.Mekhan.,45,No.2,376-380,1981 to the case of the follower load. Linearized expressions for their determination were obtained in /5–8/ in accordance with the theory of small deformations. The theory of finite deformations /9/ was used in /1,2/ to obtain the expressions which led to a number of the qualitative and quantitative results /1,2,10,11/ within the framework of three-dimensional, linearized theory of stability for the case of small subcritical deformations, which coincide with the corresponding results of the three-dimensional linearized theory of stability for the case of finite subcritical deformations. As we know, the latter theory is free of the errors of kinematic character, and this confirms the expediency of using the refined expressions of /1,2/. In what follows, we shall also use, for the sake of generality, the expressions given in /5–8/, calling the latter the normal, and those of /1,2/ the refined expressions.

The contravariant components $q^j$ of the surface load written for the problem in question in the Lagrangian coordinates have, according to the refined expressions /1,2/, the form

$$q^j = -q\,(N^j \nabla_\alpha u^\alpha - N^\alpha g^{\beta j} \nabla_\beta u^\alpha) \tag{1.3}$$

while, according to the normal expressions /5–8/ we have

$$q^j = -qN^\beta \nabla_\beta u^j \tag{1.4}$$

Here $N^j$ denote the contravariant components of the unit normal to the surface under a specified follower load, in the natural (undeformed) state, and $g^{i\beta}$ denote the contravariant components of the metric tensor in the natural state. The covariant differentiation is carried out with help of the basis vectors in the natural state.

Let us employ the Kirchhoff–Love hypothesis. Since the shells are thin-walled, we can assume that the pressure in the form of a follower load is applied at the middle surface at $z = 0$. In this case the expressions corresponding to the refined /1,2/ and normal /5–8/ approach have the form

$$q_x = -q\frac{\partial w}{\partial x}, \quad q_y = -q\frac{\partial w}{\partial y}, \quad q_z = q\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{w}{R}\right) \tag{1.5}$$

$$q_x = -q\frac{\partial w}{\partial x}, \quad q_y = -q\frac{\partial w}{\partial y}, \quad q_z = 0 \tag{1.6}$$

In accordance with (1.5), (1.6) and (1.1), we write the basic system of equations in the form

$$\mathbf{LU} + \mathbf{BU} + \mathbf{\Pi U} - \rho h \mathbf{QU}^{\cdot\cdot} = 0 \tag{1.7}$$

The term $\mathbf{\Pi U}$ corresponds to the perturbation in the surface load where $\mathbf{\Pi}$ is the matrix of differential operators, the nonzero elements of which have the form

$$\begin{gathered}\Pi_{ij} = \Lambda \Pi_{ij}^*,\ \Lambda = q\,(1-\nu^2)/(Eh),\ \Pi_{13}^* = -\delta_1 \partial/\partial x \\ \Pi_{22}^* = -\delta_1/R,\ \Pi_{23}^* = -\delta_1 \partial/\partial y,\ \Pi_{31}^* = -\delta_2 \partial/\partial x \\ \Pi_{32}^* = -\delta_2\, \partial/\partial y,\ \Pi_{33}^* = \delta_2/R\end{gathered} \tag{1.8}$$

Various values of $\delta_i\ (i = 1, 2)$ correspond to particular cases of the problem in question. When $\delta_1 \equiv \delta_2 = 0$, the external pressure appears in the form of a dead load. The case $\delta_1 = 1,\ \delta_2 = 0$ corresponds to external pressure in the form of a follower load, the components of which are obtained from the normal /5–8/ expressions. When $\delta_1 \equiv \delta_2 = 1$, the external pressure is reduced to a follower load the components of which are found from the refined expressions /1, 2/.

From (1.8) it follows that a symmetric matrix of differential operators under an external pressure in the form of a follower load can only be obtained in the case of the refined theory /1,2/. We also note that the results regarding the construction of the matrix $\mathbf{\Pi}^*$ remain in force also for the inelastic models of the shell material. Next we shall investigate the case of a hinged shell in the subcritical membrane state.

**2. Hinged shell.** In the case of a support hinged along the end faces, the boundary conditions at $x = 0$ and $x = L$ have the following form:

$$w = 0, \quad v = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} = 0 \tag{2.1}$$

We shall assume that the end faces of the shell rest on supporting frames which ensure that a subcritical membrane state exists /4/. In this case the elements of the matrix $\mathbf{B}$ have the form (1.2). The first condition of the hinged support (2.1) at the end faces ensures the fulfilment of the first sufficient condition of applicability (formula (9) of /12/) of the Euler method of investigating the problem in question. In this connection we shall delete from (1.7) the inertial terms and thus obtain an eigenvalue problem with boundary conditions (2.1) and the equation

$$\mathbf{AU} = 0 \tag{2.2}$$
$$A_{11} = \frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial y^2}, \quad A_{12} \equiv A_{21} = \frac{1+\nu}{2}\frac{\partial^2}{\partial x\,\partial y},$$
$$A_{13} = -\left(\frac{\nu}{R} + \delta_1\Lambda\right)\frac{\partial}{\partial x}$$
$$A_{22} = \left(1 + \frac{h^2}{12R^2}\right)\left(\frac{\partial^2}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x^2}\right) - \frac{\delta_1}{R}\Lambda,$$
$$A_{23} = -\left(\frac{1}{R} + \delta_1\Lambda - \frac{h^2}{12R}\Delta\right)\frac{\partial}{\partial y}$$
$$A_{31} = -\left(\frac{\nu}{R} + \delta_2\Lambda\right)\frac{\partial}{\partial x}, \quad A_{32} = -\left(\frac{1}{R} + \delta_2\Lambda - \frac{h^2}{12R}\Delta\right)\frac{\partial}{\partial y}$$
$$A_{33} = \frac{h^2}{12}\Delta^2 + \frac{1}{R^2} + R\Lambda\frac{\partial^2}{\partial y^2} + \delta_2\frac{\Lambda}{R}, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

We note that under the action of a follower load the matrix $\mathbf{A}$ will be symmetric only when the refined expressions /1,2/ are used, while (2.2) implies that the matrix $\mathbf{A}$ will not be symmetrical when the normal expressions /5 – 8/ are used. At the same time, as follows from /12/, the problem in question for the hinged shell will be self-conjugate. Below we shall investigate the problem, in a general form, for all three particular cases.

We write the displacements satisfying the boundary conditions (2.1) at $x = 0$ and $x = L$ in the following form:

$$u = f_1\cos\frac{m\pi x}{L}\sin\frac{ny}{R}, \quad v = f_2\sin\frac{m\pi x}{L}\cos\frac{ny}{R} \tag{2.3}$$
$$w = f_3\sin\frac{m\pi x}{L}\sin\frac{ny}{R}, \quad f_i = \mathrm{const}$$

From (2.2) and (2.3) we obtain the characteristic equation in the form

$$\det\|\alpha_{ij}\| = 0; \; i, j = 1, 2, 3 \tag{2.4}$$
$$\alpha_{11} = \frac{1-\nu}{2} + \varepsilon, \quad \alpha_{12} \equiv \alpha_{21} = \frac{1+\nu}{2}\sqrt{\varepsilon}, \quad \alpha_{13} = (\nu + \delta_1 p)\sqrt{\varepsilon}$$
$$\alpha_{31} = (\nu + \delta_2 p)\sqrt{\varepsilon}$$
$$\alpha_{22} = (1+\gamma)\left(1 + \frac{1-\nu}{2}\varepsilon\right) + \delta_1 p n^{-2}, \quad \alpha_{23} = 1 + \delta_1 p + \gamma n^2(1+\varepsilon)$$
$$\alpha_{32} = 1 + \delta_2 p + \gamma n^2(1+\varepsilon), \; \alpha_{33} = n^4\gamma\,(1+\varepsilon)^2 + 1 - n^2 p + \delta_2 p$$
$$\varepsilon = \left(\frac{m\pi R}{nL}\right)^2, \quad \gamma = \frac{h^2}{12R^2}, \quad p = R\Lambda$$

If we compute the elements of the matrix $\mathbf{L}$ using the technical theory of shells only, then we should delete from the elements $\alpha_{23}$ and $\alpha_{32}$ of the determinant (2.4) the terms containing the multiplying factor $\gamma$.

## 3. Analysis of the characteristic equation.

We shall carry out an analysis of the characteristic equation (2.4) corresponding to the technical theory of shells. Let us construct solutions of (2.4) for particular cases in the form of series in terms of small parameters. Let us consider the case when the following conditions hold:

$$\gamma \ll 1, \; \varepsilon \ll 1 \tag{3.1}$$

From (2.4) it follows that the first condition of (3.1) holds always for the thin-walled shells. The second condition of (3.1) demands that the number of waves along the directrix be greater than the number of waves along the generatrix. Since in the course of experimental investigations a single half-wave is always formed along the generatrix and several waves along the directrix, we can assume that the second condition of (3.1) also holds. In this case we shall seek a solution of (2.4) in the form

$$p = \sum_{i,j=0}^{\infty} x_{ij}\varepsilon^i\gamma^j \tag{3.2}$$

Assuming that $O(\varepsilon^3) \ll \gamma \lesssim O(\varepsilon^2)$, we shall limit ourselves to the approximation

$$p \approx x_{00} + x_{10}\varepsilon + x_{20}\varepsilon^2 + x_{01}\gamma \tag{3.3}$$

From (2.4) and (3.3) we obtain two roots for $x_{00}$

$$(x_{00})_1 = 0, \quad (x_{00})_2 = -\delta_1\frac{n^2 + \delta_1(1 - n^{-2})}{1 + \delta_2(1 - n^{-2})} \tag{3.4}$$

In the case of a follower load, the second root (3.4) has no physical meaning since in this case $(x_{00})_2 < 0$. According to (1.4), (3.3) and (3.4), we have

$$(x_{10})_1 = 0, \quad (x_{20})_1 = \frac{1-\nu^2}{n^2+\delta_1(1-n^{-2})}, \quad (x_{01})_1 = \frac{n^4}{n^2+\delta_1(1-n^{-2})} \tag{3.5}$$

From (3.3)—(3.5) we obtain the following expression:

$$p \approx \left(n^2\gamma + \varepsilon^2\frac{1-\nu^2}{n^2}\right)\frac{n^4}{n^4+\delta_1(n^2-1)} \tag{3.6}$$

Setting in (3.6) $\delta_1 \equiv \delta_2 = 0$, we obtain the value of the root for the case when the external pressure has the form of a dead load

$$p \approx n^2\gamma + \varepsilon^2\frac{1-\nu^2}{n^2} \tag{3.7}$$

The expression (3.7) coincides with the known result (/4/, p.496) provided that the condition (3.1) holds, and this implies that the values of the cirtical load also coincide.

When the external pressure has the form of a follower load, we put in (3.6) $\delta_1 = 1$ and $\delta_2 = 0$ or $\delta_1 \equiv \delta_2 = 1$ to obtain the value of the root with the accuracy of (3.3), in the following form:

$$p \approx \left(n^2\gamma + \varepsilon^2\frac{1-\nu^2}{n^2}\right)\frac{n^4}{n^4+n^2-1} \tag{3.8}$$

We note that, within the approximation given by (3.1) and (3.3), in the case of the external pressure in the form of a follower load, the value of the root obtained using the refined approach /1,2/ $(\delta_1 \equiv \delta_2 = 1)$ coincies with the value of the root obtained using the normal approach /5—8/ $(\delta_1 = 1, \delta_2 = 0)$ and has the form (3.8). However, in contrast to the normal approach, the refined approach yields a symmetric matrix of differential operators of the self-conjugate problem. In the general case the quantitative differences between the solutions of the problem with external pressure in the form of a follower and dead load can only be explained by applying numerical methods to the equation (2.4).

In conclusion, we shall show under which conditions the solutions of the problems with external pressure in the form of the follower and dead loads coincide. From (2.4), (3.1), (3.7) and (3.8), it follows that the solutions in question become identical when the following conditions hold:

$$(m\Pi R/nL)^2 \ll 1, \; n^2 \gg 1 \tag{3.9}$$

Thus we see that if the shell parameters are such that conditions (3.9) hold for the classical solution /3/ (see also /4/) in the case of a dead load, then the solution also holds when the external pressure has the form of a follower load. In this case a comparison with the results of experimental study is also more correct in the case when the external pressure assumes the form of a hydrostatic load.

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